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Dedekind's Mathematical Structuralism: From Galois Theory to Numbers, Sets, and Functions

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In this essay, "mathematical structuralism" will be understood mainly as a style of work, a methodology for mathematics—but methodological choices can hardly be made without concern for the subject matter. Richard Dedekind's case was no exception to this rule. Thus his mathematical structuralism, which will be our main concern, was supplemented by a philosophical conception of mathematical objects.[1]

What is meant by "structure" in this context? Roughly, a structure is a relational system, a framework (*Fachwerk*, truss) of relations between elements—where the emphasis is on the relations (and relations of relations, etc.), in the sense that the same structure can be instanced by different kinds of elements. This rough sketch can be elaborated in a number of different ways, both mathematically and philosophically.

What, then, do we mean by "mathematical structuralism"? It is a style of work that takes results in a given branch of mathematics to emerge from the nature of relevant structures (exemplified therein), and often typically, from certain interrelations between structures of different kinds. A clear and paradigmatic example, also for Dedekind, is Galois theory, as we will see.

The essay will proceed as follows: After some background on Dedekind's main forerunners (§1), we will consider structuralist themes in his approach to Galois theory and algebraic number theory (§2). Then we will turn to his rethinking of the real numbers (§3) and the natural numbers (§4), within a general framework of sets and functions. The essay will end with a brief summary and conclusion (§5).

[1] The way in which we use "mathematical structuralism" in this essay makes it closely related to "methodological structuralism" in Reck and Price (2000); cf. the editorial introduction to this volume. We also use "style" in a methodological and epistemological sense, as opposed to a personal, national, or merely aesthetic one; cf. Mancosu (2017) for a general discussion.

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1. Forerunners: Gauss, Dirichlet, and Riemann

As just indicated, a core ingredient of mathematical structuralism is the emphasis on relations, as opposed to objects standing in those relations. Henri Poincaré is well known for having written in *Science and Hypothesis*: "Mathematicians do not study objects, but the relations between objects; to them it is a matter of indifference if these objects are replaced by others, provided that the relations do not change" (Poincaré [1902] 2011, 20). Less well known is the fact that he said this as a preparation for explaining Richard Dedekind's account of the real numbers as defined by cuts.[2] Yet this point of view has deeper roots, also reaching further back than Dedekind.

By 1900, a structuralist approach was natural for many mathematicians, especially those, like Poincaré, used to working with group theory; similarly for Hilbert and mathematicians influenced by his application of the axiomatic method to geometry. But already in the 1820s, C. F. Gauss had argued that "mathematics is, in the most general sense, the science of relations" (Gauss [1917] 1981, 396).[3] This is so since "the mathematician abstracts entirely from the nature of the objects and the content of their relations; he is concerned solely with counting and comparison of the relations among themselves" (Gauss [1831] 1863, 176).[4] In another pregnant remark, he wrote that some mathematical results should be obtained "from notions [i.e., concepts], not from notations" (quoted in Dedekind 1895, 54). At the same time, Gauss's style of doing mathematics was still mostly classical; and while he took care to reformulate some existing theories in terms of pregnant "notions" (such as the congruence relation, ≡, in number theory), his writings often seem more calculational than structural.

Around 1850, several German mathematicians insisted that one ought to "put thoughts in the place of calculations", as Dirichlet wrote in his obituary of Jacobi. In other words, they adopted the principle—later attributed by Hermann Minkowski to Dirichlet himself—of obtaining mathematical results with a "minimum of blind calculation, a maximum of clear-seeing thought" (quoted in Stein 1988, 241). And by the end of the 19th century it had become customary to speak of a *conceptual approach* to mathematics in this connection, as opposed to more calculational approaches.[5] Riemann and Dedekind,

[2] See the essay on Poincaré in the present volume for more.

[3] Our translation; in the original German: "Die Mathematik ist so im allgemeinsten Sinne die Wissenschaft der Verhältnisse [in der] man von allem Inhalt der Verhältnisse abstrahiert" (Gauss [1917] 1981, 396).

[4] Our translation; in the original German: "Der Mathematiker abstrahirt gänzlich von der Beschaffenheit der Gegenstände und dem Inhalt ihrer Relationen; er hat es bloss mit der Abzählung und Vergleichung der Relationen unter sich zu thun" (Gauss 1831, 176).

[5] For more on the opposition between a "conceptual" and a more "computational" approach to mathematics, cf. Stein (1988), Laugwitz (2008), also Tappenden (2006), Reck (2016).

two young mathematicians influenced directly by Dirichlet, adopted this principle wholeheartedly. They also gave it a particularly abstract twist, or as one might say, a philosophical bent.

The initial model in this connection was Dirichlet's work from the 1830s, specifically his contributions to analytic number theory and the theory of trigonometric series. Gustav Lejeune Dirichlet is not as well known today as he deserves; but his mathematical results were "jewels" (Gauss in an 1845 letter to Humboldt)[6] that greatly influenced the development of mathematics. Moreover, his lectures—recorded, edited, and published by Dedekind—were highly influential and celebrated for their conceptual clarity. When he proved a theorem, one would never get lost in a jungle of calculations; instead, one would come away with clear insight into the chain of reasons, into the crucial steps that make the result possible. In addition, in Dirichlet's work on Fourier series (1829) he promoted analysis with more rigor than Cauchy. He was able to prove the existence of a Fourier-series representation for any function that is continuous and does not oscillate too often. Crucially, this result *necessitated* a "conceptual approach," since the goal was to establish the existence of a series representation merely from some very general traits of functions.

Dirichlet's application of methods from analysis to pure number theory (1837) was also greeted as an impressive novelty. The first example was his theorem that there are infinitely many primes of the form $a + n \cdot b$, with a and b coprime. The key point here is that recourse to certain functions in analysis (called L-series) was presented as indispensable; thus a result about finite numbers could only be obtained via a detour through the infinitesimal calculus. This stimulated much thought about the foundations of mathematics, especially by Kronecker and Dedekind. Dirichlet's own conclusion seems to have been that *pure mathematics is just arithmetic*, i.e., that all of analysis and algebra is nothing but a heavily developed number theory. Thus, as Dedekind later recalled, in the 1850s he often heard Dirichlet say that any result of algebra or analysis, no matter how complex or apparently remote, could be reformulated purely as a theorem about the natural numbers ([1888a] 1963a, 35). This would, among others, justify the application of analytic methods to number theory in a deep way, implying that there is nothing "impure" in it.

One more aspect of these contributions by Dirichlet is crucial for our purposes. It is his conceptual approach to mathematics that led him to emphasize the idea of an *arbitrary function*. Up to then, a "function" was supposed to be given explicitly by means of a formula, say polynomial or a concrete infinite

[6] And "one does not weigh jewels on a grocer's scales" (Biermann 1977, 88).

series. However, Dirichlet defined a function to be a "law" according to which "to any x there corresponds a single finite y", i.e., an arbitrary correspondence of numerical values (Ferreirós 1999, 148). A function f is then *continuous* if small variations of x correspond to small variations of $f(x)$. Assuming now that, within an interval, the function f is bounded, is continuous except in finitely many points, and has finitely many maxima and minima, Dirichlet established that there is a Fourier-series representation for it.

A general way to understand this result is that the notion of function representable by a Fourier series, which makes it "calculational", is tantamount to a notion defined more abstractly or *conceptually*, namely that of a piecewise continuous, piecewise monotone function f. This is how Bernhard Riemann presented the matter in the introduction to his PhD thesis on the theory of analytic (complex-valued) functions. As such, Dirichlet's approach constitutes a substantial triumph for the conceptual style of thinking. Riemann then made it his programmatic goal to base the theory of *complex* functions on a similarly conceptual starting point, leaving the development of explicit "forms of representation" for the very end of the treatment. Here is how he characterized the resulting methodological perspective:

> Previous methods of treating these functions were always based on an expression for the function, taken as its definition, which determined its value for *each* value of the argument. Our investigation has shown that, as a consequence of the general characteristics of [analytic] functions of a complex variable, in such a definition a part of the determining elements follows from the rest; and the extension of those determining elements has been reduced to what is strictly necessary. This simplifies their treatment considerably. To give an example, in order to establish the equality of two different expressions for the same function, it was necessary to transform one into the other, that is, to show that they coincided for each value of the variable magnitude; now it is sufficient to show their coincidence in a far more restricted domain.
>
> A theory of such functions in accordance with the foundations established here would determine the configuration of the function (that is, its value for each value of the argument) independently of forms of determination by means of operations; to the general concept of a[n analytic] function of a complex magnitude, one would only add the necessary traits for determining the function, and only afterwards would one move on to the different expressions which the function admits. What is common to a species of functions that have been expressed in a similar way by means of operations would then be represented by means of boundary and discontinuity conditions. (Riemann [1851] 1876, §20, 38–39; our trans.)

The basis for Riemann's approach was his definition of an *analytic function* via the Cauchy-Riemann conditions,[7] together with his study of functions by means of their associated Riemann surfaces, plus some additional conditions regarding points of discontinuity (poles and singularities) and boundary conditions.[8]

The association of a *Riemann surface*—a geometric or, better, topological object—with each analytic function was a very fruitful move too, but one that remained somewhat mysterious at the time. In retrospect it can be regarded as another step toward mathematical structuralism: the study of one kind of object (a complex function) by associating it with an object of a different kind (a surface in n-dimensional space). The price paid by Riemann and his followers was foundational worries concerning the nature of these novel objects, which required the development of n-dimensional geometry and topology in order to be fully resolved. Finally, applying the same kind of methodology to the study of Euclidean space, Riemann subsumed the latter under the much richer and quite abstract concept of *continuous (and differentiable) manifold*, endowed with a certain metric.[9] The idea here was to look for further conditions so as to gradually narrow the scope of spaces falling under this general concept, thereby clarifying the nature of the assumptions behind Euclidean geometry and its links to other recently developed geometries, like the non-Euclidean one of Lobatchevsky-Bolyai, or even more generally, to geometries in spaces of variable curvature (cf. Ferreirós 2006).

In Riemann's work, the conceptual style of doing mathematics became very explicit and exclusive. As a consequence, it was criticized by other mathematicians—most importantly by Weierstrass and his Berlin school—who wanted to remain closer to the previous concrete and constructive style of mathematics.[10] In particular, Weierstrass gave preference to explicit representations of functions by means of power series. He argued, among others, that the class of differentiable functions had not been characterized completely yet (constructively, as one should add); and along such lines, the definition of analytic functions given by the Cauchy-Riemann conditions was not entirely satisfactory. For Dedekind, in contrast, the example of Riemann's style of mathematics became the model to emulate. Thus, when Dedekind makes his most committed

[7] These conditions say, in essence, that a function is *analytic* or holomorphic if and only if it is differentiable (in the complex domain); they state: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ for $u + vi = f(x + yi)$.

[8] Thus, if the function has discontinuities only in isolated points, and they consist in its "becoming infinite with finite order," then the function "is necessarily algebraic" and vice versa.

[9] Cf. Scholz (1999). An n-dimensional manifold is currently defined as a topological space that, locally, behaves like Euclidean space—but globally it won't in general be like $\mathbb{R}^n$. Riemann introduced the idea in connection with his reflections about n-dimensional geometry: they generalized the idea of a 2-dimensional surface to three and more dimensions. The Riemann surfaces are 2-dimensional manifolds, and it may be impossible to embed them in Euclidean space.

[10] Cf. Bottazzini and Gray (2013), 320–324, and Tappenden (2006), 108–122.

statements about mathematical method, there is typically a reference to Riemann involved, as in the following example:

> In these last words, if they are taken in their most general sense, we find the expression of a great scientific thought: the decision for the inner in contrast to the outer. This contrast also comes up in almost all fields of mathematics. One only has to think of function theory, of Riemann's definition of functions by means of characteristic inner properties, from which the outer forms of representation arise with necessity. But in the much more limited and simple field of ideal theory too, both directions have their validity. (Dedekind 1895, 54–55)[11]

Similarly, in the preface to his 1871 book on algebraic number theory, which contains his first presentation of ideal theory, Dedekind expressed his "hope that the effort to obtain characteristic basic concepts [*das Streben nach characteristischen Grundbegriffen*], which has been crowned with such beautiful success in other areas of mathematics, may not have eluded me completely" (Dedekind 1930–32, 3:396–397, our trans.). The same outlook is presented in a letter to Lipschitz from 1876, again with reference to Riemann.[12]

Dedekind was exposed to Dirichlet's and Riemann's conceptual style of thought during his time as privatdozent at the University of Göttingen. This proved to be a crucial experience for him. Not only did he later repeat Dirichlet's view that all of algebra and analysis is an extended form of arithmetic, as already noted; he also adopted his general notion of function (with consequences we explore further subsequently). And whenever it came to expressing his most deeply cherished methodological preferences, he wrote that his aim (in algebra, in number theory, etc.), like Riemann's in his theory of functions, was to base his results on "characteristic concepts," while letting concrete "forms of representation" emerge only as end products.

Actually, in Dedekind's hands the consistent promotion of such goals led even further—to a form of *mathematical structuralism*. This Dedekindian move brought with it novel set- and map-theoretic methods. But before we turn to those, the historical roots of yet another core ingredient of his mathematical

[11] Our translation; in the original: "In diesen letzten Worten liegt, wenn sie im allgemeinsten Sinn genommen werden, der Ausspruch eines großen wissenschaftlichen Gedankens, die Entscheidung für das Innerliche im Gegensatz zu dem Äußerlichen. Dieser Gegensatz wiederholt sich auch in der Mathematik auf fast allen Gebieten; man denke nur an die Funktionentheorie, an Riemanns Definition der Funktionen durch innerliche charakteristische Eigenschaften, aus welchen die äußerlichen Darstellungsformen mit Notwendigkeit entspringen. Aber auch auf dem bei weitem enger begrenzten und einfacheren Gebiet der Idealtheorie kommen beide Richtungen zur Geltung."

[12] From the letter to Litschitz (in our trans.): "My efforts in the theory of numbers are directed . . . —though this comparison may sound pretentious—to attain in this field something similar to what Riemann did in the field of function theory" (Dedekind 1930–32, 3:468, our trans.). For additional remarks, compare, e.g., p. 296.

structuralism should be made explicit, namely: the systematic exploitation of relations, not just between particular mathematical objects, but between whole systems of objects as exemplified by Galois theory.

2. The Algebraic Context: From Galois Theory to Algebraic Number Theory

B. L. van der Waerden, the author of the classic textbook *Moderne Algebra* (1930), stated: "Galois and Dedekind are those who gave modern algebra its structure—the supporting skeleton of this structure comes from them" (1964, vii). Let us consider what he meant by that, including how Galois' and Dedekind's approaches are related. After that, we will turn to Dedekind's closely related use of Galois theory in his work on algebraic number theory.

After having finished his dissertation under Gauss in 1852, Dedekind remained in Göttingen as a privatdozent for six more years. He hesitated about what to do next. He also attended several of Dirichlet's and Riemann's classes so as to broaden and deepen his knowledge of mathematics. In 1855, he found his first great field of work: the contributions of Abel and Galois to higher algebra, into which he immersed himself, and especially, Galois's theory (first published, posthumously, in 1846 and quite difficult to understand at the time). In 1856–57 and 1857–58, Dedekind gave the first university courses in Germany on Galois theory. And it is here that he started to develop "the structural and conceptual methodology that will be characteristic of his whole mathematical work" (Scharlau 1981, 336, our trans.).

As is well known, algebra had been understood as the general theory of the symbolic resolution of equations for centuries; or as Isaac Newton put it, it was a kind of "universal arithmetic" that worked with a symbolic or literal calculus (in German: *Buchstabenrechnung*) instead of ordinary arithmetical calculations. It was primarily Galois' work in the early 19th century that led to a novel, much more abstract understanding of algebra—later often called "modern algebra"—in which the resolution of equations is relegated to the level of applications, while issues involving general theories of groups and fields come to the forefront (see Corry 2004, chap. 1). Dedekind played a crucial role in that development.

A central algebraic problem, from the 15th to the 19th century, was to find general methods for solving polynomials of any degree by means of radicals—just as the second-degree equation $ax^2 + bx + c = 0$ is solved by taking $x = (-b \pm \sqrt{(b^2 - 4ac)})/2a)$. Analogous resolutions were found for equations of degree 3 and 4 in the 16th century. But around 1800 mathematicians were convinced that a general solution, for all degrees n, is impossible to obtain. Lagrange, Ruffini, and Abel provided increasingly fine-grained analyses of this question, leading to Abel's proof that equations of degree 5 are in general not solvable (by radicals). This line of mathematics emphasized analyzing *permutations* of the roots of the equation at issue, and some expressions that remain *invariant* under such permutations. The very young Galois picked up on that approach, noting that all the permutations together form a *group*—a very innovative and rather abstract concept. This led him to associate with each equation its "Galois group" *G*, and then to investigate subgroups of *G* with particular attention to what later (by Heinrich Weber) would be called "normal" subgroups, which proved to be crucial.

Rather quickly, Dedekind obtained remarkable clarity in rethinking Galois' crucial innovation. Here is one passage in which he explains the path he took:

> During my first in-depth study of [the Gaussian theory of] cyclotomy[13] during the Pentecost holidays of 1855, I had, while well understanding all the details, to fight long and hard until I found the crucial principle in irreducibility; I only had to direct simple, natural questions at it so as to be led, with necessity, to all the details. Through a careful study of the algebraic investigations of *Abel* and, especially, *Galois*, and by my discovery, in early December of the same year, of the most general relation between any two irreducible equations, these thoughts were brought to a certain conclusion. Later I employed the method I had found also in the two winter courses on cyclotomy and higher algebra [given at Göttingen] in 1856–58. (Dedekind 1930–32, 3:414–415)[14]

Dedekind's lecture notes from these courses were only published, by Wilfried Scharlau, in the 1980s. In Scharlau's evaluation, his presentation of Galois theory—with its group-theoretic and field-theoretic foundations (see below in this section)—was far ahead of his time, even satisfying 20th-century expectations (Scharlau 1981, 341).[15] A similar level would only be achieved again in

[13] Cyclotomy is the study of roots of equations of the form $x^m = 1$, with m a positive integer. These roots are points on the unit circle (and thus cut it, "cyclotomy").

[14] Our translation; in the original German: "Bei meinem ersten gründlichen Studium der Kreisteilung in den Pfingstferien 1855 hatte ich, obgleich ich das Einzelne wohl verstand, doch lange zu kämpfen, bis ich in der Irreduktibilität das Prinzip erkannte, an welches ich nur einfache, naturgemäße Fragen zu richten brauchte, um zu allen Einzelheiten mit Notwendigkeit getrieben zu werden. Nachdem diese Gedanken durch eine eingehende Beschäftigung mit den algebraischen Untersuchungen von *Abel* und namentlich von *Galois* vervollständigt und durch die im Anfang Dezember desselben Jahres gelungene Auffindung der allgemeinsten Beziehungen zwischen irgend zwei irreduktiblen Gleichungen zu einem gewissen Abschluß gekommen waren, habe ich später in meinen beiden Wintervorlesungen über Kreisteilung und höhere Algebra 1956–1958 die damals gewonnene Methode befolgt."

[15] Dedekind's version of Galois theory was also much superior to contemporary ones, e.g., those by Betti or Serret (or Galois himself). It is comparable to the (often celebrated) Jordan (1870), but may be again superior to it as a presentation of the theory as a whole.

Heinrich Weber's *Lehrbuch der Algebra* (1895) and in Dedekind's Supplement XI to Dirichlet's *Vorlesungen über Zahlentheorie* (1894).[16]

Two ingredients of Galois theory and Dedekind's reception of it are of special importance for our purposes: the group-theoretic aspect of Galois's original contribution, developed further by Dedekind; and the introduction of the concept of a field. The latter was only implicit, thus still obscure, in Galois's writings, while Dedekind made it explicit and very central. Concerning the former, in his 1857–58 lectures Dedekind presents very clearly a theory of finite groups, which he already understands in a general, abstract way. Thus he writes:

> The following investigations are based solely on the two fundamental theorems just proven,[17] together with the fact that the number of substitutions is finite. Hence its results will be valid equally for *any domain* with a finite number of *elements, things, concepts* $\theta, \theta', \theta'', \ldots$ that admits of a *composition* $\theta\theta'$, from θ and θ', which is defined arbitrarily but so that $\theta\theta'$ is again a member of that domain and this kind of composition obeys the *laws* expressed in both fundamental theorems. In many parts of mathematics, but especially in number theory and algebra, we repeatedly find examples of this theory; and the same methods of proof are valid there too. (Scharlau 1981, 63, emphasis added)[18]

The structuralist flavor of this passage is undeniable. It is also not hard to see that the two theorems or laws mentioned suffice to axiomatize finite group theory. Dedekind then adds the idea of partitioning a group by a normal subgroup, with an induced law of composition. All of this is quite remarkable for the 1850s.

Dedekind introduces the notion of a field initially under the label "rational domain" (*rationales Gebiet*). The insight that, when studying an algebraic equation, one has to pay attention to the domain of numbers in which its coefficients

[16] One of the students attending the courses, Paul Bachmann, remarked about Dedekind: "In his calmly flowing, never halting presentation, [he was able to] present the theory with such exceptional clarity and simplicity that it was not hard for me to comprehend the material, then still quite foreign to me, despite its abstractness—the concept of group played a big role" (our trans.). In the original German: Dedekind was able "in ruhig fliessendem, niemals stockenden Vortrage die Theorien mit so ausnehmender Klarheit und Einfachheit [vorzutragen], dass es mir nicht schwer wurde, den mir damals noch ganz fremden Gegenstand trotz seiner Abstraktheit—der Gruppenbegriff spielte eine grosse Rolle—verständnisvoll zu erfassen" (quoted in Landau 1917, 53).

[17] The theorems in question state the associativity of the product, and a law of simplification: from any two of the three equations $\phi = \theta$, $\phi' = \theta'$, $\phi\phi' = \theta\theta'$, the third follows.

[18] In the original German: "Die nun folgenden Untersuchungen beruhen lediglich auf den beiden soeben bewiesenen Fundamentalsätzen und darauf, dass die Anzahl der Substitutionen endlich ist: Die Resultate derselben werden deshalb genau ebenso für ein Gebiet von einer endlichen Anzahl von Elementen, Dingen, Begriffen $\theta, \theta', \theta'', \ldots$ gelten, die eine irgendwie definierte Composition $\theta\theta'$ aus θ und θ' zulassen, in der Weise, dass $\theta\theta'$ wieder ein Glied dieses Gebietes ist, and dass diese Art der Composition den Gesetzen gehorcht, welche in den beiden Fundamentalsätzen ausgesprochen sind. In vielen Theilen der Mathematik, namentlich aber in der Zahlentheorie und Algebra, finden sich fortwährend Beispiele zu dieser Theorie; dieselben Methoden der Beweise gelten hier wie dort."

live, together with the domain containing its roots (regarded as different from the first), was due to Galois. The way in which he introduced them was by considering rational functions of given quantities supposed to be "known *a priori*"; as he writes: "We shall call *rational* any quantity which can be expressed as a rational function of the coefficients of the equation and of a certain number of *adjoined* quantities arbitrarily agreed upon" (quoted in Toti Rigatelli 1996, 119). Galois was not more explicit than that—but it was now relatively easy for mathematicians like Dedekind, or Kronecker, to go further and explicitly define fields. This can be done in different ways, and it is instructive to compare the contrasting styles involved.

We mentioned earlier that Weierstrass wanted to remain close to a "premodern", concrete, and calculational style of mathematics. The same applies, all the more, to Kronecker. He essentially followed Galois in defining a "domain of rationality" (*Rationalitätsbereich*) as the totality of quantities that are rational functions of some given quantities $r', r'', r''', \ldots$. Kronecker was explicit in preferring this kind of (constructivist) approach, via explicit expressions, to its more abstract alternative. Dedekind, in contrast, chose to emphasize the link between the notion of a "field" (*Körper*)—as he came to call it around 1870—and the "simplest arithmetic principles" (Dedekind 1930–32, 3:400). Thus, he defined a field as a set of numbers "closed in itself" under addition, subtraction, multiplication, and division. In doing so, he was *directly avoiding* any reliance on explicit expressions for numbers, since this would "spoil" (*verunzieren*) the presentation.

These two definitions are closely related but not exactly equivalent. Kronecker's "domains of rationality" are always engendered by *finitely* many elements $r', r'', r''', \ldots$, while Dedekind's "rational domains" or "fields" do not face such a restriction. As a consequence, the totality of *all* algebraic numbers is a Dedekindian field, but not a Kroneckerian domain of rationality; similarly for the field $\mathbb{R}$ of all real numbers, which was not accepted by Kronecker at all. Moreover, in Dedekind's treatment of what he called a "finite field", i.e., a finite extension of $\mathbb{Q}$, he was not happy with the definition that it is the extension of $\mathbb{Q}$ obtained by adjoining a number α, i.e., the set $\mathbb{Q}[\alpha]$ of all numbers $x_0 + x_1\alpha + x_2\alpha^2 + \ldots + x_{n-1}\alpha^{n-1}$ with coefficients $x_i \in \mathbb{Q}$. Instead, he preferred to call K a "finite field" over $\mathbb{Q}$ when there are only a finite number of subfields K' such that $\mathbb{Q} \subseteq K' \subseteq K$. This is again a *conceptual* definition. It is also one that directly points to an invariant property, as Dedekind was well aware (see Ferreirós 1999, 94). And again, explicit equations or "forms of representation" are relegated to being auxiliary means.

The contrast between the very different methodologies involved—Kronecker's constructivist approach and the conceptual/structural approach of Dedekind—became even clearer and more explicit in their divergent ways of dealing with ideal theory (or the "theory of divisors" in Kronecker's terminology). We will say more about the latter soon. But the style of Dedekind's work is already visible in general traits of his approach to Galois theory. Note, in addition, that Dedekind focuses on the basic foundations of the *whole theory*, i.e., on what we would call its structural underpinnings. In doing so, he relegated the study of concrete solutions of equations to a secondary role, thereby also departing from Galois.[19] What he was mainly concerned about was a general understanding of the *existence* and *nature* of such solutions, not concrete processes of solution.

It remains to highlight one further aspect of the shift from Galois to Dedekind. From today's point of view, the key moves in Galois theory are the following: (i) we associate with a given equation its Galois group G, so as to investigate its subgroups; (ii) we note that there is a correspondence between the subgroups of G and intermediate fields K (intermediate between the base field B, where the coefficients of the equation lie, and its extension E, containing all the roots of the equation); and (iii) we investigate the conditions for obtaining the splitting field E (as a finite extension of B) by studying the properties of the subgroups of G.[20] Galois introduced aspect (i), while (ii) and (iii) were added, and well understood, by Dedekind already in the 1850s. They also illustrate an element of mathematical structuralism we take to be central. Namely, a structuralist methodology often involves addressing *problems about certain structures* by studying their *interrelations with other structures*, perhaps of a *different kind*; and these structural correspondences may require the introduction of *novel objects* along the way.[21] We would like to highlight this aspect especially, since it is often ignored or at least underemphasized by philosophers of mathematics in discussing structuralism.

During the 1860s, a period in which Dedekind moved from Göttingen to Zürich for his first salaried position and then back to his hometown of Braunschweig as professor, he came to view the concept of a number field as the central object of study for algebra. This was consistent with the arithmetizing orientation he had encountered in Dirichlet's work, which guided his research on pure mathematics from early on (like that of several other mathematicians at the time: Weierstrass, Cantor, etc.). To provide outsiders at least with a rough sketch of this conception of algebra, he wrote in 1873 that it deals with the "algebraic [family] relations between numbers" or, better, that it is "the science of [family] relations between fields" (Dedekind 1930–32, 3:409).[22] In particular, the

[19] Fragments of Galois's writings that were oriented more toward this question included details not given by Dedekind (e.g., about irreducible equations of prime degree); cf. Scharlau (1981, 107).

[20] For a classic presentation of Galois theory along such lines, cf. Artin (1942).

[21] Concerning the latter, cf. the introduction of Riemann surfaces. Concerning the former, this amounts to studying relevant morphisms and functors (in category-theoretic language).

[22] Our translation; in the original German: The new algebra deals "von den algebraischen Verwandtschaften der Zahlen"; it is "die Wissenschaft von der Verwandtschaft der Körper."

properties of equations studied both traditionally and in Galois theory can be reconceived as properties of fields and their interrelations (base field, splitting field), as previously noted.

As Scharlau remarked (1981, 106), Dedekind was close to publishing the first textbook of "modern algebra", with a careful redaction of his 1856–58 notes on Galois theory. He failed to do so only because he found "an even more interesting" field of work in algebraic number theory, to which he then directed most of his energies. The exact date of the redaction at issue is not fully clear, but it seems safe to assume that it must have been finished by 1860, if not earlier. In any case, the structure of Dedekind's carefully written notes is distinctive and instructive. Its first section contains an investigation of the group-theoretic results needed in Galois-theoretic algebra; the concept of a (finite) group is isolated and investigated separately; and both are given an abstract, fully general presentation.

What is characteristic here, and a constant in Dedekind's subsequent writings, is this: while investigating a given area of mathematics, he was always on the lookout for *new concepts* that might be useful; and when he became convinced that a certain new idea was needed, he would isolate it and develop its *general theory* separately. As another example, his 1877 presentation of ideal theory begins with a section entitled "Auxiliary Theorems from the Theory of Modules" (in which he introduces an antecedent of the more general 20th-century concept of R-module, where R is a ring);[23] and in all later presentations, the theory of modules forms a section of its own, rising to a rather central role in his 1894 version of ideal theory.

Galois theory remained important for Dedekind's work in algebraic number theory. His first approach to the latter was in terms of a combination of the principles of Galois with a theory of "higher congruences" (Dedekind 1930–32, 3:397).[24] Algebraic numbers are those numbers (real or complex) that are roots of a polynomial with rational coefficients, e.g., $\sqrt{-3}$ (root of $x^2 + 3$) or $\sqrt{1+\sqrt{5}}$ (root of $x^4 - 2x^2 - 4$). Now, in certain simple cases it was clear at the time which numbers should be regarded as algebraic *integers* in such contexts, e.g., numbers of the form $a + b\sqrt{3}$, with a, b integers.[25] But in general the situation was not so clear. Both Dedekind and Kronecker considered this issue; and each of them was helped by previous acquaintance with the concept of a field or "rational domain". Each realized that one has to go to the relevant field first, so as then to isolate the ring of integers in it (to use current terminology). As a consequence both hit on the right definition of an algebraic integer, namely a number (real or complex)

[23] Dedekind's "modules" are in fact $\mathbb{Z}$-modules, where $\mathbb{Z}$ is the usual ring of integers.

[24] Meant are polynomial congruences modulo a prime; cf. Haubrich (1992, chap. 8).

[25] Adjoining $\sqrt{3}$ to $\mathbb{Q}$, we obtain a number-field, denoted $\mathbb{Q}[\sqrt{3}]$, that is a finite extension of $\mathbb{Q}$. The numbers specified are the integers corresponding to that field.

that is the root of a *monic* polynomial with integral coefficients ($\sqrt{1+\sqrt{5}}$ is an example).[26]

When studying the number theory of certain algebraic integers and building on the cases treated by Gauss earlier, Ernst Kummer had found the following problem: one often ends up in a situation in which *prime* integers do not conform to our expectations. Dedekind later gives this simple example: In the domain of integers $\mathbb{Z}[\sqrt{-5}]$, the numbers 2, 3, $1+\sqrt{-5}$ and $1-\sqrt{-5}$ are indecomposable, i.e., they are not the product of two other integers of this kind. However, they do not behave like regular primes, for $2 \cdot 3 = (1+\sqrt{-5}) \cdot (1-\sqrt{-5}) = 6$, i.e., *unique decomposability* of integers into prime factors *fails*.[27] Kummer then had the brilliant idea of introducing "ideal numbers," objects that do not exist in the given domain of integers, but that, once assumed, allow us to recover the principle of unique decomposition.

The main issue in algebraic number theory on which Dedekind was working, from the 1860s on, was to develop an analysis of all the domains of algebraic integers in which the fundamental principle of unique decomposition holds. The core question became how to define Kummer's "ideal numbers" in a way that could be applied to any ring of integers and that was rigorous, e.g., by allowing us to introduce the product operation on them carefully and explicitly. Around 1860 he worked with a theory based on "higher congruences," as already mentioned, which led him close to that goal. However, he was not fully satisfied with this approach, both since it was not completely general and since it was not sufficiently conceptual. The key to his eventual success, 10 years later, was the *extensionalization* of the whole problem, in the sense of its analysis in *set-theoretic* terms. As he put it himself:

> I did not arrive at a general theory . . . until I abandoned the old, more formal approach completely and replaced it by another, one that departs from the simplest basic conception and fixes the eyes directly on the end. In that approach, new creations are not needed any more, like those of Kummer's ideal number. It is entirely sufficient to consider systems of really existing numbers, which I call ideals. The power of this notion rests on its extreme simplicity. (Dedekind 1877, 268, our trans.)[28]

[26] Monic means that the lead coefficient of the polynomial is 1, as happens in the case of $x^4 - 2x^2 - 4$.

[27] This refers to the *Fundamental Theorem* of number theory, due to Gauss, which holds for the regular integers (in **Z**) as well as for the Gaussian integers $a + bi$ (with $i = \sqrt{-1}$).

[28] Our translation; in the original French: "Je ne suit parvenu à la théorie générale . . . qu'après avoir entièrement abandonné l'ancienne marche plus formelle, et l'avoir remplacée par une autre partant de la conception fondamentale la plus simple, et fixant le regard immédiatement sur le but. Dans cette marche, je n'ai plus besoin d'aucune création nouvelle, comme celle du nombre idéal du Kummer, et il suffit complétement de la considération de ce système de nombres réellement existants, que j'appelle un idéal. La puissance de ce concept reposant sur son extrême simplicité."

That is to say, instead of considering an ideal number *p* in Kummer's sense, which was only a fiction introduced formally, Dedekind considers the totality of algebraic integers in the given ring divisible by *p*—which forms an infinite set. This set is called an *ideal A*. In some cases (those of principal ideals), it corresponds to a number in the ring that divides all the elements of *A*, but not in other cases. For Dedekind the task now became to find a simple definition of such ideals *A*; and he found that two conditions suffice: (i) sums and differences of elements of *A* are again elements of it; (ii) the products of elements of *A* with any integers in the ring are again in *A*. His new definition worked fully generally; and he proceeded to treat ideals (infinite sets) as if they were simple numbers, operating on them as "new arithmetical elements". Doing so allowed him to define the product of ideals; it also made possible the proof of the fundamental theorem for any ring of algebraic integers.

We will not go into further details concerning Dedekind's theory of ideals, since it has been analyzed extensively elsewhere.[29] But two general observations are worth adding in our context. First, the downside of Dedekind's success with his conceptual, set-theoretic, and structuralist techniques was that others at the time were puzzled by his very "abstract" moves. Those moves were natural for him, but foreign to most mathematicians of that generation. Consequently, his ideal theory was not accepted until the 1890s; and even then, David Hilbert, Adolf Hurwitz, and others preferred more formal approaches.[30] As late as 1917, Edmund Landau would remark that in a "modern lecture" aiming to prove the main results, without gaps but briefly, one would prefer Hurwitz's approach, and "Dedekind's definition of an ideal is not used as basic any more [*wird kaum noch zu Grunde gelegt*]" (Landau 1917, 59).

The merit of Hurwitz's more formalistic way was that it avoided "the long chain of classical concepts and theorems of Dedekind's, about field permutations [automorphisms], modules, modules of rang *n*, etc." (Landau 1917, 59). Dedekind published a paper on methodology (1895) in which he explained why his self-contained approach was to be preferred to the Hilbert-Hurwitz way of relying on established algebraic theories. But his structuralist methodology, exemplified by his contributions to Galois theory and algebraic number theory, only came into vogue in the 1920s and later, with works by Emmy Noether, Emil Artin, B. L. van der Waerden, etc.[31] Thus the "modern algebra" of the 1920s would take Dedekind's side—whence Noether's well-known phrase, "It's all in Dedekind already."

[29] Cf. Avigad (2006), earlier Edwards (1980) and Ferreirós (1999, 95–107).

[30] Hurvitz took inspiration from Kronecker's use of polynomial rings and the "method of indeterminates" (*Methode der Unbestimmten*). Hilbert followed that style in his famous *Zahlbericht*, which made it much less structuralist than Dedekind's work (see the introduction to Hilbert [1897] 1998).

[31] Cf. Corry (2004), as well as the essays on Noether, Bourbaki, and Mac Lane in this volume.

The second general observation is that similar "abstract" moves, which again elicited negative reactions, characterize Dedekind's contributions to more foundational issues, as we will see next. The latter also led him to a kind of logicism.

3. The Real Numbers: From Arithmetization to Dedekindian Logicism

Early in the 20th century, Charles Sanders Peirce called Dedekind, very aptly, a "philosophical mathematician." Or to quote him more fully: "The philosophical mathematician, Dr. Richard Dedekind, holds mathematics to be a branch of logic" (Peirce [1902] 2010, 32). Dedekind's logicism was developed in the context of reconceptualizing first the real and then the natural numbers. But it is illuminating to go back further, to Dedekind's earliest foundational reflections.

Dedekind's is a singular case in the history of mathematics, in our judgment, because of the intensity and the success with which he devoted himself to reshaping his discipline. Indeed, he worked on a systematic reshaping of all the "pure mathematics" of his time—arithmetic, algebra, analysis—a fact that has not been recognized enough so far.[32] In doing so, he set the stage for various 20th-century developments—by being a key precursor of Hilbert, Bourbaki, and, above all, "modern algebra". From the beginning of his career, Dedekind was deeply concerned about foundational issues in mathematics as well. In fact, foundational and more mainstream issues were intimately intertwined for him.

Dedekind's interest in foundations is already apparent in his habilitation lecture, whose topic was "the introduction of new functions in mathematics" (Dedekind 1854). In this lecture, he proposed a genetic viewpoint on the number systems, one according to which "the requirement of the unrestricted possibility of carrying through the indirect or inverse operations [subtraction, division, etc.] leads with necessity to the creation of new classes of numbers" (quoted in Ferreirós 1999, 218). However, the set-theoretic considerations typical of his later writings were not present in this discussion yet, which focused on how to redefine such operations rigorously and non-arbitrarily in expanded domains (e.g., how to extend the arithmetic operations from the positive and negative integers to the rational numbers). On the other hand, it is noteworthy that Dedekind speaks of new kinds of numbers as our "creations" already in this context. He also believed that the main difficulties in systematizing arithmetic begin with the imaginary numbers (Dedekind 1930–32, 3:434).

[32] For the rise of "pure mathematics" in this sense, including Gauss's role, cf. Ferreirós (2007).

Interestingly, the latter is an issue to which he would never contribute. The reason seems to be that he found a completely satisfactory solution just a few years later, while reading W. R. Hamilton.[33] Here the ordered pair $<a, b>$ is not yet conceived as a set—but we are moving in that direction. In all likelihood, Dedekind was completely satisfied with this *reduction* of complex arithmetic to the arithmetic of the real numbers. And the same move became then a central part of his foundational project: to reduce expanded number-domains, together with their operations and laws, to simpler ones. The quintessential example—and a key advancement for the foundations of mathematics—can be found already in 1858, with Dedekind's new approach to the real numbers. But its results were only published in 1872, in his well-known essay *Stetigkeit und irrationale Zahlen*.[34]

As the details of this episode are again well known (or easy to find in the literature),[35] we will only highlight the core ideas. Dedekind starts by assuming that the arithmetic of the rational numbers $\mathbb{Q}$ (an ordered field) has been satisfactorily developed. His goal is to introduce "new arithmetic elements"—the irrationals—*in one step*, as a whole system. By only presupposing $\mathbb{Q}$, he thus reduces the newly created numbers (and their operations) to the rational numbers. In particular, Dedekind proves all the fundamental properties of the new number domain $\mathbb{R}$ based on the operations on and properties of the rationals: his 1872 essay contains a proof that $\mathbb{R}$ is an *ordered* field with the (topological) property of *continuity* or, in later terminology, *line-completeness.* An essential proviso, however, is this: Dedekind needs to regard as unproblematic that we can work set-theoretically with the totality of rational numbers—the *reduction* of $\mathbb{R}$ to $\mathbb{Q}$ is *by set-theoretic means*.

The key in Dedekind's approach to the real numbers is his concept of a *cut*: a Dedekind-cut $<A_1, A_2>$ on $\mathbb{Q}$ is a pair of (non-empty) sets A_1, A_2 such that each element of A_1 is *less than* any element of A_2, i.e., $\forall x \in A_1 \, \forall y \in A_2 \, (x < y)$. Crucially for him, cuts on the system of rational numbers are a "purely arithmetical phenomenon" (Dedekind [1888a] 1963, 35–36, 40). By presupposing as given also the *totality* of all Dedekind-cuts for the number-system $\mathbb{Q}$, we have

[33] Cf. Ferreirós (1999, 220–221). Hamilton, in the introduction to *Lectures on Quaternions* (1853), defined the complex numbers $a + bi$ as ordered pairs of real numbers $<a, b>$, including corresponding operations. In manuscripts by Dedekind from the 1860s, perhaps earlier, he defines the integers as pairs of natural numbers and the rationals as pairs of integers; cf. Sieg and Schlimm (2005).

[34] Dedekind started teaching the calculus at the University of Zürich in 1858. It is in that context that he came up with his theory of cuts; cf. Dedekind (1872, 1), and Dedekind (1888a, 36). (In the English translation of the latter, 1853 is wrongly given as the relevant year).

[35] Besides the original Dedekind (1872), see, e.g., Courant and Robbins (1996, 71–72), Ebbinghaus et al. (1983, 30–31), or earlier Landau (1930, chap. 3). Dedekind was not the only mathematician working on this topic at the time, as mentioned by these writers; but his approach to it was quite original.

essentially introduced *the real number system* in its entirety—some cuts will correspond to rational numbers, while others will not, e.g., the cut $A_1 = \{x: x^2 < 3\}$, $A_2 = \{x: x^2 > 3\}$. It is by means of the latter that the irrationals numbers are introduced.

Dedekind's other central contribution in this context is his masterful definition of *continuity*: a set of elements S endowed with an ordering < is continuous if and only if, given a corresponding cut of its elements into two (non-empty) classes C_1 and C_2 (as defined previously), there exists *one and only one* element c_0 of S that "produces" it. (This definition presupposes implicitly that S is a *densely ordered* set, a point that gave rise to some debate and misunderstandings at the time. Also relevant is the fact that Dedekind continuity implies the Archimedean property.)[36] A straight line in geometry, with an ordering of its points left-and-right, intuitively has the mentioned property: for any cut, there is a point that produces it.[37] As Dedekind established explicitly, the system of all cuts on $\mathbb{Q}$ has the property too.

Using the concept of a *field isomorphism*—present already in Dirichlet (1871), a year before the publication of *Stetigkeit und irrationale Zahlen*[38]—his procedure for introducing the system of real numbers can then be described as follows: $\mathbb{R}$ is defined as a *novel* number system isomorphic to the system of cuts on $\mathbb{Q}$. More specifically, we "create new numbers" corresponding to all the cuts, including those not produced by rational numbers, and together these form the system $\mathbb{R}$. The arithmetic properties of the real numbers, thus introduced, are derived rigorously from the arithmetic of the rational numbers; similarly for a linear ordering on $\mathbb{R}$, induced by that of $\mathbb{Q}$. And $\mathbb{R}$ can now be shown to be *continuous* in the precise sense introduced earlier (just like the system of cuts on $\mathbb{Q}$).

As emphasized already, (infinitary) set theory is functioning as a key background assumption in Dedekind's foundational work (also in his work in algebra and algebraic number theory). But how did Dedekind understand its status? Consider again his view that cuts are a "purely arithmetic phenomenon." Underlying it is the assumption that set theory is *pure logic*; and hence, set-theoretic constructions on $\mathbb{Q}$ are *pure arithmetic*, since we are allowed to employ all of logic's resources in it. It is on this basis that the phenomenon of cuts

[36] This says that any positive number r, multiplied by itself n times, will be greater than any other number s. The Archimedean property excludes *infinitesimal* numbers.

[37] In the introduction to Dedekind (1888a) he points out, however, that we can conceive of a geometric space that does not have this property, e.g., $\mathbf{A}^3$ where $\mathbf{A}$ is the set of algebraic numbers. This is relevant for evaluating Euclid's traditional approach to geometry.

[38] The label "isomophism" is not Dedekind's, however. In 1871, he spoke of a field *substitution* (*Substitution*) instead. The term "isomorphism" was employed early on in crystallography; it was also used in Jordan (1870, 56) for groups. Compare http://jeff560.tripod.com/i.html.

appears "in its logical purity" according to him ([1888a] 1963, 40). Notice also, once again, that together with the set ℚ of rational numbers Dedekind assumes as given the totality of all cuts on ℚ—a strong assumption equivalent to (an application of) Zermelo's power set axiom.

While controversial today, the idea that the concept of set is purely logical was common during Dedekind's time, e.g., in the tradition of the algebra of logic from Boole onward (cf. Ferreirós 1996; 1999, 47–53). Dedekind adopted this view early on, it seems, and it formed a key ingredient in his promotion of an early form of *logicism*. Already in a manuscript drafted in 1872, the same year in which his essay on the real numbers was published, he introduced sets in general as follows: "A *thing* is any object of our thought. . . . A *system* or *collection* [*Inbegriff*] *S* of things is determined when for any thing it is possible to judge whether it belongs to the system or not" (Dugac 1976, 293, our trans.). And in 1887, while preparing the final version of his essay on the natural numbers, he noted that the theory of sets, or "systems of elements," is "logic" (quoted in Ferreirós 1999, 225).

Because Dedekind regarded set theory as pure logic, the fact that the theory of the real numbers can be reduced to the arithmetic of the rational numbers by set-theoretic means implied for him that the notion of the continuum *does not* have to be seen as grounded in perception or geometric intuition. As he puts it, the number concept is "entirely independent of the intuitions of space and time" (Dedekind [1888a], 1963, 31); and the creation of the "pure, continuous number domain" (ℝ) is not dependent on the notion of magnitude. Instead, its creation takes the form of "a finite system of simple steps of thought" (340), and we get a "purely logical construction" (*Aufbau*) of arithmetic—in the broad sense, from ℕ to ℝ, or even to the field ℂ of complex numbers.

Clearly the set-theoretic reduction of the irrationals to more elementary number systems was a crucial step for Dedekind. It also seems that he was the first mathematician to consciously avoid reliance on the traditional notion of magnitude in this context (cf. Epple 2003). A further reason for this avoidance was a *requirement of purity*. As he wrote: "I demand that arithmetic shall be developed out of itself" (Dedekind [1872] 1963, 10) and, more particularly, "without any admixture of foreign ideas (such as that of measurable magnitudes)" ([1888a] 1963 35, trans. modified). Again, Dedekind's initial goal—delineated already in 1854, clarified while reading Hamilton, and encouraged by Dirichlet's approach—was to develop the complex number system starting from the natural numbers. Other contributors to "arithmetization," like Weierstrass, shared this goal; but unlike them, Dedekind realized this could be done with the help of set theory alone. Arithmetic is thus shown to be an outgrowth of the "pure laws of thought" (Dedekind [1882] 1963, 31).

Dedekind's version of logicism was highly influential during the 1890s—much more so than Frege's—by affecting authors such as Schröder and Hilbert.[39] The Peirce quotation given at the beginning of this section reflects this state of affairs. On the other hand, Dedekind's talk of "creation" has often been taken to throw doubts on the alleged *logical* nature of his point of view. And it has to be conceded that his way of expressing things sometimes runs the risk of conflating logic and psychology.[40] Was he then guilty of a problematic form of psychologism (as later criticized by Frege and Husserl)?[41] Dedekind was always convinced that mathematical objects and concepts are our "creations"—in his eyes, the prototype objects are numbers, and these are "free creations [*freie Schöpfungen*] of the human mind" ([1888a] 1963, 35; [1872] 1963, 4; also 1854). This was perhaps his most persistent philosophical conviction, from 1854 until his death.[42] Yet such talk about "the human mind" does not have to be understood in a subjectivist sense, as psychologistic thinkers are usually assumed to do. Instead, it can be interpreted in a Kantian or neo-Kantian way; it can thus be seen as a reference to our collective "mind" and its products, thus to human cognition and culture.[43] And as we will see in the next section, by 1888 the "creation" of the natural numbers consists merely in a step of *abstraction* from a more concrete "simply infinite set," so that strictly logico-mathematical results *determine* every single aspect of arithmetic.[44]

One final observation concerning the real numbers: how Dedekind proceeds in this context is closely related to his approach to *ideal theory*—methodologically the two are *of a piece*. Indeed, in a French essay of 1877 he explicitly compares the two cases (Dedekind 1877, 268–269). In both, we introduce new "arithmetical elements" in the progressive expansion of the number systems (although Dedekind does not "create" new objects corresponding to his set-theoretic ideals). And in both he is guided by the following desiderata: (1) "Arithmetic ought to be developed out of itself" ([1872] 1963, 10, trans. modified), thus avoiding any "foreign elements" and "auxiliary means" (magnitudes in the case of the reals, polynomials or other specific representations in the case of ideals).

39 Cf. Ferreirós (2009), later also Reck (2013a).

40 The same happens in Schröder's logical writings. And traces of it are still visible in Hilbert, e.g., when he writes: "We think [*wir denken*] of three sets [*Systeme*] of things" (Hilbert 1930, 2); similarly in his paper on the real numbers: "We think of a set of things [*Wir denken ein System von Dingen*]" (Hilbert 1900, 181). Notice the use of Dedekind's terminology in both cases.

41 Cf. Reck (2013b) for related charges, as well as Dedekind's more general reception.

42 In a letter to Weber of 1888, he wrote that we have the right to claim for ourselves such a creative power: "We are of divine lineage and there is no doubt that we possess creative power, not only in material things (railways, telegraphs), but quite specially in mental things" (Dedekind 1888b).

43 Cf. the use of *Geisteswissenschaften* in German, later often translated as "cultural sciences." Many 19th-century philosophers were intensely concerned about them.

44 Note also that, despite his frequent talk of "construction," Dedekind's basic tendency is not at all constructivistic (in the technical sense). As his theory of the real numbers shows, it is classical and objectivistic, just like Frege's. More on the underlying set theory in the next section.

(2) When new elements are introduced, they must be defined in terms of operations and laws found in the previously given domains (the arithmetic of $\mathbb{C}$ in the case of ideals).[45] (3) The new definitions must be completely general, applying "invariantly" to all relevant cases (we should not define some irrationals as roots, others as logarithms, etc.; we should not employ different means when determining ideal factors in various cases, as Kummer had done). (4) The definitions must offer a solid foundation for the deductive structure of the whole theory; they ought to be not just sound definitions, but the basis for rigorous proofs for all relevant theorems.

These four desiderata are closely related to Dedekind's mathematical structuralism, especially (3) and (4). Moreover, they guide his approach to the natural numbers too, as we will see in the next section.

4. Natural Numbers, Sets, and Functions: Logicism Systematized

While working on Galois theory and algebraic number theory, Dedekind distills out the core concepts of group and field, so as then to investigate them further abstractly and generally (similarly for the concepts of ideal, module, and, in later work, lattice). When developing his theory of the real numbers, his approach is similarly *conceptual*. The concept of field is again crucial in this context, but also that of continuity, defined in terms of cuts. Importantly, these concepts all involve global properties, which affects entire systems of objects—they are "structural" in that sense. We noted earlier that mathematical structuralism typically also involves the study of interrelations between such systems. This too is true for Dedekind's approach to the reals. Not only is an *isomorphism* (for ordered fields) between the system of cuts and that of the real numbers involved, at least implicitly;[46] his domain extension from $\mathbb{Q}$ to $\mathbb{R}$ also brings with it a corresponding *homomorphism*, as he is well aware. And while more heuristic than formally rigorous, his comparison of the reals with the intuitive geometric line involves such an interrelation too.

Dedekind's approach to the natural numbers in his 1888 essay displays the same general features; but there are also some noteworthy changes. In his approach

[45] This requirement was particularly critical at the time. Today we usually treat number systems axiomatically, but this is done (explicitly or implicitly) within the framework of set theory.

[46] Similarly, Dedekind acknowledges an isomorphism between his system of cuts and the reals constructed via (equivalence classes of) Cauchy sequences, as Cantor, Méray, etc. proposed. This is implicit in his remark (letter to Lipschitz, July 27, 1876) that Cantor and Heine have achieved the same goals as himself (reduction to the rational numbers, establishment of the continuity property), and that their expositions are different "only externally". See also Sieg and Schlimm (2017).

to algebra, algebraic number theory, and analysis, Dedekind always deals with subsets of the complex numbers (and related operations and functions). When dealing with the natural numbers, in contrast, he starts to consider sets (*Systeme*) of objects in *complete generality*. As he writes: "It very frequently happens that different things . . . can be considered from a common point of view, can be associated in the mind, and we say that they form a *system S*". Moreover, the concept of thing involved here is very inclusive: "I understand by *thing* every object of our thought" (Dedekind [1888a] 1963, 44). The other crucial aspect about sets *S* is that their identity is now understood *extensionally*—all that matters is that "it is determined with respect to every thing whether it is an element of *S* or not" ([1988a] 1963, 45). In a footnote Dedekind adds that a decision procedure is not required in this connection, thereby distancing himself from Kronecker. Clearly his notion of set is classical, not constructivist.

Parallel to this generalized notion of set, Dedekind introduces a generalized notion of function—or "mapping" (*Abbildung*). In his own words again: "By a *mapping Φ* of a system *S* we understand a law according to which to every determinate element *s* of *S* there *belongs* a determinate thing called the image of *s* and denoted $\Phi(s)$" (Dedekind [1888a] 1963, 50, trans. modified).[47] As Dedekind's use of the term "law" in this passage indicates, he is consciously building on Dirichlet's notion of function, while also broadening it even further (from an arbitrary functional correlation between sets of numbers to one between any two sets of objects). And unlike in axiomatic set theory, he does not reduce functions to sets of tuples; for him the notions of set and function are equally basic. Indeed, both belong to *pure logic*, in line with our earlier discussion. At a few points, Dedekind even seems to suggest that the notion of function or mapping is the really basic one.[48]

What Dedekind proposes in his 1888 essay is, thus, a general *logicist framework* in which to reconstruct arithmetic (from ℂ all the way down). However, he does not formulate basic laws or axioms for it (as Frege was quick to point out).[49] Instead, he applies it in his reconstruction of the natural number sequence, i.e., in reducing the latter to logic. The core concept here is that of a *simply infinite system* (*einfach unendliches System*) which involves the concept of *infinity* for sets. Famously, a set *S* is (Dedekind-)infinite if it can be mapped 1-1 onto a proper subset of itself (Dedekind [188a] 1963, 63). A set *N* is simply infinite if,

[47] In W. W. Beman's translation (1963) of Dedekind (1888), *Abbildung* is rendered as "transformation," which seems awkward and is less appropriate than "mapping".

[48] As Dedekind writes, he was led to it by scrutinizing counting and numbers. It constitutes "an ability without which no thinking is possible"; and in particular, the entire science of numbers is built "upon this unique and in any event absolutely indispensable foundation" (Dedekind 1963, 32). He does not write anything as strong about the notion of set; compare Ferreirós (2017).

[49] Cf. Reck (2019), also for Dedekind's relation to Frege more generally.

in effect, there is an element *a* in *N* and a 1-1 function *f* on *N* such that *N* = {*a*, *f*(*a*), *f*(*f*(*a*)), . . .}. More rigorously and formally, Dedekind's definition of being simply infinite involves four conditions, including one that uses the abstract concept of a "chain" to express a minimality condition on the set *N*, thus guaranteeing induction for simply infinite systems. [50] It is not hard to see that these four conditions constitute a (more abstract) variant of the Peano axioms—or better, the *Dedekind-Peano axioms*.

Dedekind's reconstruction of the natural numbers is again well known, so that we will only survey some highlights here (cf. Reck 2003). Important for him is to establish both the existence of a simply infinite system and (what we would call) the categoricity of that notion—the fact that any two simple infinities are isomorphic. In a well-known letter to Keferstein (Dedekind 1890), he clarifies that the former is meant to ensure the consistency of the notion of simple infinity. And with his categoricity theorem, Dedekind makes explicit an aspect not present yet in his earlier treatment of the reals. (Any two continuous ordered fields are isomorphic too, but this was not proved in 1872.) In addition, categoricity implies, as noted in passing, that exactly the same theorems hold for all simply infinite systems: i.e., the Dedekind-Peano axioms are semantically complete.[51] Finally, a careful justification for proofs by mathematical induction and for definitions by recursion is provided.

There are two controversial parts of Dedekind's 1888 essay. First, his (attempted) proof for the existence of a simply infinite system, which proceeds *via* arguing that an infinite system exists, relies on a universal set, which makes it fall prey to Russell's antinomy.[52] Second, Dedekind includes the following additional step not mentioned so far: start with a simply infinite system (any of them will do, since they are all isomorphic); then "neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another" ([1888a] 1963, 68) It is exactly at this point in his essay that Dedekind adds: "With reference to this freeing the elements from every other content (*abstraction*) we are justified in calling numbers a *free creation of the human mind*" (68, emphasis added). However, it is not obvious how to interpret Dedekind's appeal to "abstraction" and "free creation," especially in a non-psychologistic way.

[50] Modernizing his notation slightly, the four conditions are the following: Consider a set *S* and a subset *N* of *S* (possibly equal to *S*). *N* is said to be *simply infinite* if there exists a function *f* on *S* and an element *a* in *N* such that (i) *f* maps *N* into itself; (ii) *N* is the minimal closure of {*a*} under *f* in *S*; (iii) *a* is not in the image of *N* under *f*; and (iv) *f* is a 1-1 function. (Dedekind uses the notion of "chain" in (ii), to capture what it means to be the minimal closure of a set under a function.)

[51] Compare Awodey and Reck (2002a), also for a discussion of the history of these notions.

[52] Dedekind appeals to "the totality of things that can be objects of my thought" (1888a, 64). This may again sound psychologistic, but is meant objectively; cf. Klev (2018).

In correspondence from the same year, Dedekind makes clear that he takes his introduction of "the natural numbers" in his 1888 essay to be exactly parallel to his introduction of "the real numbers" in 1872, although the "abstraction" aspect has now been made more explicit (cf. Dedekind 1888b). Also, in both cases all resulting theorems are determined—entirely objectively—by the basic concepts involved, in the sense that it is determined what holds for any system of objects falling under them.[53] Beyond that, there are two interpretations of "Dedekind abstraction" that have been proposed in the literature. According to the first, a *novel* simply infinite system is introduced by it, a system isomorphic to but not identical with the one we started with and, in addition, determined "purely structurally." According to the second interpretation, such abstraction merely amounts to treating the *original* simple infinity in a certain way, namely by identifying it pragmatically as "the natural numbers," with the proviso that any other simple infinity could play the same role. [54] The case of the reals, or of continuous ordered fields, is parallel.

This essay is not the place to decide which interpretation of "Dedekind abstraction" is more defensible.[55] But with either one of them, we have arrived at a structuralist conception of *mathematical objects* that complements mathematical structuralism in the *methodological* sense; the latter leads to the former in Dedekind's writings, i.e., mathematical structuralism to philosophical structuralism. Turning our attention back to mathematical structuralism, note that, besides Dedekind's continued "conceptualism", the consideration of structure-preserving mappings (morphisms) between different systems of objects has become central in his foundational writings. This is most explicit in the categoricity theorem from his 1888 essay, which involves isomorphisms between any two simply infinite systems. A more implicit case is the treatment of recursive definitions and proofs by induction in it, which relates the natural number sequence to other recursively generated systems in terms of corresponding homomorphisms.[56]

By 1888, Dedekind has come to rely on a general framework of sets and functions for his mathematical structuralism. But as already noted, he does not formulate basic laws or axioms for it. There are some indications that implicitly

[53] As Dedekind writes: "The relations or laws, which are derived entirely from the conditions α, β, γ, δ in (71) are therefore always the same in all ordered simply infinite systems" (1963, 68). (For those conditions, see note 50.)

[54] The first interpretation amounts to reading Dedekind as a "non-eliminative structuralist," while the second amounts to reading him in an "eliminative" way; cf. Reck and Price (2000).

[55] A decision based on Dedekind (1888a) alone may be impossible; both sides can appeal to evidence in it. For the first reading, cf. Reck (2003); for the second, Sieg and Morris (2018). Dedekind may also have moved from one position to the other, i.e., changed his mind in this connection.

[56] From the perspective of category theory, Dedekind's procedure points toward thinking of **N** in terms of a corresponding universal mapping property; cf. McLarty (1993).

he works with a naive comprehension principle for sets.[57] Because of Russell's and related antinomies, this is no longer attractive to us. What one can still do is to carefully reconstruct which more restricted set-formation principles Dedekind actually needs for his overall project. This seems, in fact, to be exactly what Zermelo did while formulating his axiomatization for set theory in 1908. In retrospect, what Dedekind needs is the following: the power set axiom and an axiom of infinity;[58] principles for set-theoretic unions, intersections, or subsets more generally (an axiom of separation); some way of introducing or reconstructing *n*-tuples; and less obviously, the axiom of choice and the axiom of replacement (missed by Zermelo originally).

As Dedekind's work brings out the importance of all these axioms, it makes sense that Zermelo, who knew the history well, considered modern set theory to have been "created by Cantor and Dedekind" (quoted in Ferreirós 1999, xii and 320). Today set theory is no longer considered to be "logic," however, among others because in its axiomatic form it is a specific mathematical theory.

5. Concluding Remarks

Our main concern in this essay has been Dedekind's mathematical structuralism, understood as a methodology or a style of doing mathematics. We can now summarize our main results briefly. From his teachers and mentors in Göttingen, especially Dirichlet and Riemann, Dedekind inherited a *conceptual* way of doing mathematics. This involves replacing complicated calculations by more transparent deductions from basic concepts. Both Dedekind's mainstream work in mathematics, such as his celebrated ideal theory, and his more foundational writings reflect that influence. Thus, he distilled out as central the concepts of group, field, continuity, infinity, and simple infinity. A related and constant aspect in his work is the attempt to characterize whole systems of objects through global properties.

From early on, Dedekind also pursued the program of the *arithmetization* of analysis—in the broad sense, from the complex numbers all the way down to the naturals. A decisive triumph came in 1858, with Dedekind's reductive treatment of the real numbers. From the 1870s on, he added a reduction of the natural numbers to a general theory of sets and mappings. This led to an early form of *logicism*, since he conceived of set theory as a central part of logic; i.e., the

[57] Or equivalently, he might work with a "dichotomy conception" where any division of the universe of objects into two parts creates corresponding sets; cf. Ferreirós (2017).

[58] Zermelo's axiom of infinity was modeled on Dedekind's controversial "proof"; he even called it "Dedekind's axiom." Its standard descendant, modified by von Neumann, still shows this origin.

reduction was ultimately to "the laws of thought." Moreover, in Dedekind's works there is a resolute reliance on the actual infinite—cuts, ideals, etc. are infinite sets. And while problematic in some respects, his attempt to execute a logicist program had a decisive effect on the rise of axiomatic set theory in the 20th century.

Its conceptualist and set-theoretic aspects are central ingredients in Dedekind's *mathematical structuralism*. But we emphasized another characteristic aspect that goes beyond both. This is the method of studying systems or structures with respect to their interrelations with other kinds of structures, and in particular, corresponding *morphisms*. A historically significant example, particularly for Dedekind, was Galois theory. As reconceived by him, in Galois theory we associate equations with certain field extensions, and we then study how to obtain those extensions in terms of the associated Galois group (introduced as a group of morphisms from the field to itself, i.e., automorphisms). Dedekind's more foundational works provide further examples, especially in terms of isomorphisms, such as his celebrated theorem that the Dedekind-Peano axioms are categorical, but also various *homomorphism* results involving the natural and real numbers.

As we saw, Dedekind connected his mathematical or methodological structuralism with a structuralist conception of mathematical objects, i.e., a form of philosophical structuralism (and the latter too involves categoricity results crucially). Central here was Dedekind's long-held view that mathematical objects, and paradigmatically numbers, are "free creations of the human mind," obtained by a kind of "abstraction" from more concrete systems of objects. With respect to Dedekind's logicism and his philosophical structuralism we acknowledged some controversial features. More can, and should, be said about both of them in the end. But we would like to conclude this essay with an observation of a different kind.

Dedekind's methodology was *not static*—it kept evolving. In fact, starting in the 1880s one can discern a subtle shift in his works, from focusing primarily on sets and set-theoretic constructions to taking functions and map-theoretic constructions as more fundamental (cf. Ferreirós 2017). However, there are only some hints to this effect in his writings, and officially both sets and functions remain basic. In addition, it was the aspects of his mathematical structuralism that we highlighted earlier with which he was most influential—on figures from Hilbert and Noether to Zermelo and Bourbaki. Finally, these aspects remain largely intact if one pushes mathematics further in a morphism-theoretic direction, as evidenced by 20th-century category theory and related developments. [59]

[59] Cf. Corry (2004), Awodey and Reck (2002b), and the essay on Mac Lane in this volume.

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